An Away-step Frank-Wolfe Method for Convex Optimization Involving a Log-Homogeneous Barrier

Renbo Zhao

Tippie College of Business
University of Iowa

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Motivating Example: D-Optimal Design

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$$i_k \in \arg\min_{i\in[m]} \nabla_i f(x^k), \qquad G_k := -\nabla_{i_k} f(x^k) - n,$$

$$j_k \in \arg\max_{j:x_j^k>0} \nabla_i f(x^k), \quad \tilde{G}_k := \nabla_{j_k} f(x^k) + n,$$

$$d^k = \begin{cases} e_{i_k} - x^k & \text{if } G_k > \tilde{G}_k \\ x^k - e_{j_k} & \text{otherwise} \end{cases}, \qquad x^{k+1} := x^k + \alpha_k d^k,$$

where the stepsize $\alpha_k \ge 0$ is given by exact line-search.

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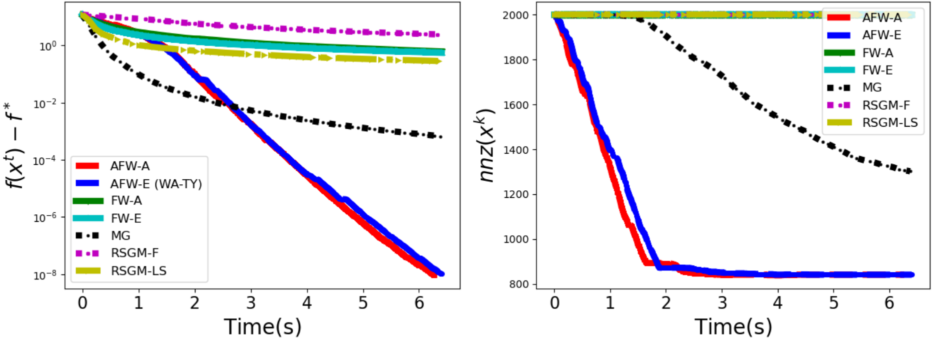
- Excellent numerical performance:



ASFW-A & ASFW-E (this work): Away-step FW methods for LHB
FW-A & FW-E [Fed72; Kha96; ZFce]: Generalized FW methods for LHB
RSGM-F & RSGM-LS [BBT17; LFN18]: Relatively smooth gradient method
MG [STT78]: Multiplicative gradient method

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- This difficulty prevents the recent analyses of the away-step FW (AFW) methods for L-smooth functions [LJJ15; BS17; PR19], as well as for *non-degenerate* generalized self-concordant function [Dvu23] being applied to (D-OPT).

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- Some deeper questions:
  - What is the essential structure of (`D-OPT`) that drives the linear convergence of the WA-TY method (or the AFW method)?
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- In this work, we will provide affirmative answers to the questions above.

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- Besides D-optimal design, other applications include
  - Budget-constrained D-optimal design
  - Positron emission tomography
  - (Reformulated) Poisson image deblurring with TV-regularization

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  1. $\left|D^3 f(y)[w,w,w]\right| \leq 2\|w\|_y^3 \quad \forall\, y \in \mathsf{int}\,\mathcal{K}, \forall\, w \in \mathbb{Y}$,
  2. $f(y_k) \to +\infty$ for any $\{y_k\}_{k\geq 1} \subseteq \mathsf{int}\,\mathcal{K}$ such that $y_k \to u \in \mathsf{bd}\,\mathcal{K}$,
  3. $f(ty) = f(y) - \theta\ln(t) \quad \forall\, y \in \mathsf{int}\,\mathcal{K}, \forall\, t > 0$.

  where $\|w\|_y := \langle \nabla^2 f(y) w, w\rangle^{1/2}$ denotes the local norm of w at $y \in \mathsf{int}\,\mathcal{K}$.

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  - (FW direction) Compute $v^k \in \arg\min_{x \in \mathcal{V}} \langle \nabla F(x^k), x \rangle$, $d^k_{\mathrm{F}} := v^k - x^k$ and $G_k := \langle -\nabla F(x^k), d^k_{\mathrm{F}} \rangle$. If $G_k = 0$, then STOP.

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  - (Choose stepsize) Choose $\alpha_k \in (0, \bar{\alpha}_k]$ in one of the following two ways:
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  - (Update iterates) Update $x^{k+1} := x^k + \alpha_k d^k$ and $\beta^{k+1} \in \Delta_{|\mathcal{V}|}$ such that $x^{k+1} = \sum_{v\in\mathcal{V}} \beta^{k+1}_v v$, and let $\mathcal{S}_{k+1} := \mathrm{supp}(\beta^{k+1})$.

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Some Remarks

Denote $\dim \mathbb{X} = n$.

- Depending on $\mathcal{X}$, we may prefer to solve $\min_{x\in\mathcal{V}} \langle \nabla F(x^k), x\rangle$ either by either minimizing over $\mathcal{X}$ (e.g., $\mathcal{X} = \prod_{i=1}^n [a_i, b_i]$) or $\mathcal{V}$ (e.g., $\mathcal{X} = \Delta_n$).
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- For all applications of interest, computing $D_k = \|\mathsf{A}d^k\|_{y^k} = \nabla^2 F(x^k)d^k d^{k\,1/2}$ takes $O(n)$ times, instead of $O(n^2)$ time.

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- ▷ $\{\delta_k\}_{k\geq 0}$ is strictly decreasing (until termination).
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$$\delta_k \leq (1-\rho)^{k_{\text{eff}}} \delta_0, \quad \text{where} \quad \rho := \min\left\{ \frac{1}{5.3(\delta_0 + \theta + B)}, \frac{\mu \Phi(\mathcal{X}, \mathcal{X}^*)^2}{42.4(\theta + B)^2} \right\},$$

where

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- All the quantities defining ρ are *affine-invariant* and *norm-independent*.

Computational Guarantees

Global linear convergence of $\{G_k\}_{k\geq 0}$:

For some (affine-invariant) $\bar{D} < +\infty$ and all $k \geq 0$, we have

$$G_k \leq \begin{cases} 4(1-\rho)^{k_{\rm eff}}\delta_0 \max\{\bar{D},1\}, & \text{if } \delta_k > 1 \\ 4\sqrt{1-\rho}^{k_{\rm eff}}\sqrt{\delta_0}\max\{\bar{D},1\}, & \text{if } \delta_k \leq 1 \end{cases}.$$

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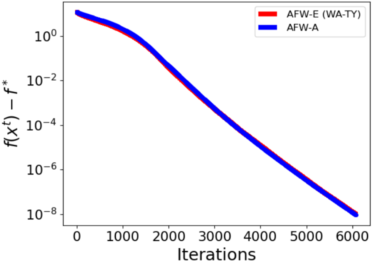

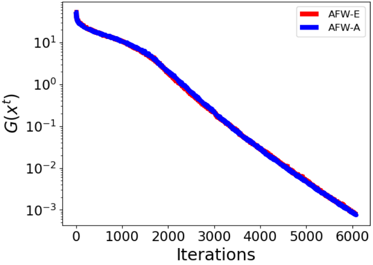

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Land on $\mathcal{F}$ in finite iterations:

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For all $k \geq \bar{k}$, if $x^k \notin \mathcal{F}$, then

- $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$, when either exact line-search or adaptive stepsize is used in Step 7,
- $\mathcal{S}_{k+1} = \mathcal{S}_k \setminus \{a^k\}$ for some $a^k \in \mathcal{S}_k \cap \bar{\mathcal{V}}_{\mathcal{F}}$, when exact line-search is used in Step 7;

otherwise, if $x^k \in \mathcal{F}$, then $x^l \in \mathcal{F}$ for all $l \geq k$.

Another Example: Positron Emission Tomography

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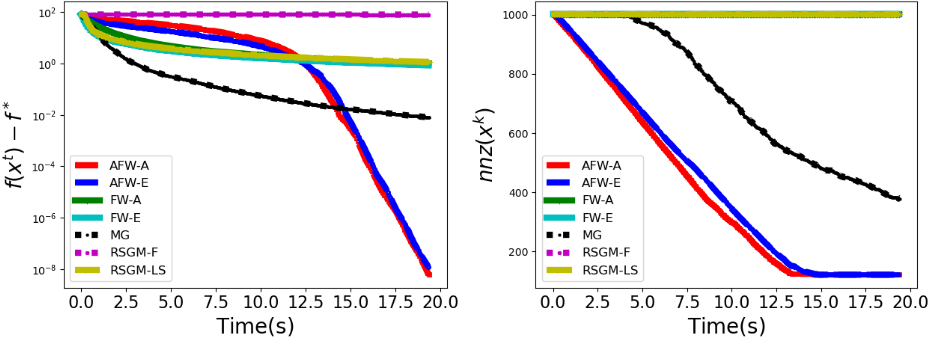

Thank you!

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